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SEQUENTIAL CONNECTIVITY OF 3-COMPLEXES

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R.H. Bing showed that if a closed 3-manifold M has a triangulation in which the 3-simplexes can be ordered $(\sigma_1, \dots, \sigma_n)$ so that each σ_1 hits $\bigcup_{j<i} \sigma_j$ in a connected set, then M is homeomorphic to S^3. There are several ways to vary Bing's idea. In the preceding case, we show that the given triangulation of $C = \bigcup_{i<n} \sigma_i$ is collapsible, so C is a 3-cell. If some of the intersections are disconnected, the sum of the numbers of excess components is a bound for the Heegaard genus of M. For more general cell complexes K (dim $\leqslant 3$), if K is collapsible then (a subdivision of) K is sequentially connected. The converse is true under certain conditions on K or on the enumeration of cells σ_i. As an application of this machine, we show that if Q is a polyhedron for which $\pi_1(Q)$ requires more than n generators, any space obtained by attaching n 2-cells to Q is not collapsible.

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Introduction

In 1951, R.H. Bing [1] showed that if a closed 3-manifold M has a triangulation T in which the 3-simplexes can be ordered $(\sigma_1, \dots, \sigma_n)$ so that each σ_i meets $\bigcup_{j<i} \sigma_j$ in a connected set then M is homeomorphic to the 3-sphere S^3. He later [2] introduced the term 'sequentially connected' to describe such a triangulation. Bing's proof involved first replacing the given triangulation with a brick partition (made of thickened vertices, thickened edges, and one remaining polyhedron in each 3-simplex) and inductively constructing a partition of S^3 isomorphic to that one. In this paper, we consider several variations on Bing's idea. First, we show that the given triangulation of the submanifold $B = \sigma_1 \cup \cdots \cup \sigma_{n-1}$ actually is collapsible. Thus, using the regular neighborhood theorem to conclude that B is a 3-cell provides an alternative proof of Bing's theorem.

There are two natural generalizations of the idea of a 3-manifold having a sequentially connected triangulation, and the proof of the above mentioned collapsing theorem readily accommodates them. (Some generalization actually is required

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to accomplish the proof.) First, we can study cell complexes (dim ≤ 3) more general than manifolds; and second, we can consider the possibility that various intersections $\sigma_i \cap \bigcup_{j<i} \sigma_j$ are disconnected. By counting the number of excess components (and including a correction term for homology of M), we obtain a numerical measure, Γ, called 'sequential connectivity', of the extent to which the given enumeration of principal cells fails to lead to a collapsing of M (to a 1-complex). By minimizing Γ over all enumerations of all triangulations of M, we obtain a numerical measure of the combinatorial complexity of M.

A standard proof of the existence of a Heegaard splitting for a 3-manifold is based on taking a regular neighborhood of the 1-skeleton of a suitably fine triangulation. By tracing through such a proof and using Theorem 3, it can be shown that, given a cell decomposition of a 3-manifold with sequential connectivity Γ, there exists a Heegaard splitting of genus Γ. Conversely, given a splitting we can construct a cell decomposition with $\Gamma = \text{genus}$. Thus sequential connectivity may be interpreted as a generalization of Heegaard genus.

The relations between combinatorial complexity and homotopy properties quickly weaken as we look in increasing dimensions. If C is a connected 1-complex, then C is collapsible iff $H_1(C) = 0$. In the case where $\dim(C) = 2$, the well known 'dunce's hat' [5] shows that C may be contractible but have no subdivision that is collapsible. In the special case where C is a 2-manifold, $H_2(C)$ gets in the way of C having a 1-spine. When $H_2(C) = 0$, a 2-manifold C always has a 1-spine appropriate to its homotopy type; so for a 2-manifold, C is collapsible iff C is acyclic. For $\dim(C) = 3$, even manifolds can behave badly as there exist [2] triangulations of a 3-cell that are not collapsible. The property of being sequentially connected may be viewed as an attempt to fit something in between being simply connected and being collapsible; having sequential connectivity $\Gamma = 0$ is slightly weaker than being sequentially connected. For a connected 1-complex, the four properties are equivalent. For connected 2- complexes, the properties are easiest to organize if we assume $H_2(C) = 0$ (since a 2-sphere is sequentially connected); then sequentially connected $\Rightarrow \not\Leftarrow \Gamma = 0 \Rightarrow \Leftarrow$ collapsible $\Rightarrow \not\Leftarrow$ contractible. For 3-manifolds where there is a chance of collapsing, i.e. connected, nonempty boundary, and trivial H_2, we have sequentially connected $\Rightarrow \Gamma = 0 \Rightarrow$ collapsible. For general 3-complexes, the reverse implications are false unless we allow subdivision; but we do not know if subdivision is necessary for cell decompositions of a 3-ball.

If a 3-complex C is collapsible, then, while C itself might not have sequential connectivity $\Gamma = 0$, a certain subdivision C^* (coarser than 2nd derived) is. This observation allows us to give a slightly different proof from Lickorish [4] of the fact that it is impossible to collapse a 3-complex formed by adding a meridional disk to a cube with a knotted hole. More generally, we show that if Q is a 3-manifold for which $\pi_1(Q)$ requires more than n generators, and C is a 3-complex obtained by attaching n disks to $\mathrm{bd}(Q)$, then $\Gamma(C) > 0$ and C is not collapsible.

The general theorem states, in part, that if C is a cell-complex of dimension ≤ 3 and there is an enumeration of the principal cells of C having sequential connectivity

Γ then we can 'engulf' the 1-skeleton of C with a 2-complex K that itself collapses to a graph of genus Γ. We would then like to find a 2-spine of C containing K. The suspension of a 'dunce hat' [5] has a triangulation with $\Gamma = 0$ but no 2-spine at all, much less than one that contains a certain collapsible subcomplex K. If C embeds in a bounded 3-manifold then any 2-complex K with a 1-spine can be enlarged to a 2-spine of C. Alternatively, if the given enumeration of principal cells of C has the property that no *three-cell* σ_i meets $\bigcup_{j<i} \sigma_j$ in *all* of $\mathrm{bd}(\sigma_i)$ then we can construct the desired 2-spine of C inductively. We have, for a number of years, been unable to determine if these alternative hypotheses can be replaced by the more appealing 'C has a 2-spine'. In particular, the following conjecture remains open: If C is sequentially connected (or, more generally, connected and $\Gamma(C) = 0$), $H_2(C) = 0$, and C has some 2-spine, then C is collapsible.

Conventions. It is convenient to use a notion of cell-complex more general than triangulations but still having the property that boundaries of cells are embedded and comprised of well defined faces. So we shall use *cell complex* to mean finite convex linear cell complex as in [3]. The *q-skeleton* of C, denoted $C^{(q)}$, is the subcomplex consisting of all cells of C having dimension $\leqslant q$. A cell is *principal* if it is not a face of any higher dimensional cell. If L is a subcomplex of C [of dimension q] and C collapses to L, denoted $C \searrow L$, then L is called a [q-] spine of C. The rank of $H_p(X; Z)$ is denoted $\beta_p(X)$. When we want to distinguish between a complex C and its underlying polyhedron, we denote the latter $|C|$.

Definition of Γ. Let $e = (\sigma_1, \ldots, \sigma_n)$ be an enumeration of the principal cells of a cell complex C. The *sequential connectivity* of e is the number

$$\begin{aligned}\Gamma(e) = &-1 + \text{number of components of } C\\ &+ \sum_{i \geqslant 2} -1 + [\text{no. of components of } \sigma_1 \cap \bigcup_{j<i} \sigma_j]\end{aligned}$$

If it happens that each intersection $\sigma_i \cap \bigcup_{j<i} \sigma_j$ is nonempty and connected, the complex C and the particular enumeration are called *sequentially connected.*

Collapsing theorems

Theorem 1 (The Very Special Case). *If an acyclic 3-manifold M has a triangulation C whose 3-simplexes can be enumerated $e = (\sigma_1, \ldots, \sigma_n)$ such that $\Gamma(e) = 0$ then C is collapsible.*

Remark. We might first note that the hypotheses tacitly state that M is compact (since C is finite), connected (acyclic), and has nonempty boundary (finite + acyclic). Also $\pi_1(M) = \{1\}$ (Van Kampen's theorem + $\Gamma = 0$) so M is contractible. Thus being able to conclude that $C \searrow 0$ is not entirely unexpected.

Proof. As the 3-simplexes of C are enumerated, we shall construct a tree $T \leqslant C^{(1)}$ and a 2-complex $K \leqslant C^{(2)}$ such that K contains $C^{(1)}$ and $K \searrow T$. It remains to show that $C \searrow K$. Add a collar $\mathrm{bd}(M) \times [0, 1)$ to M along its (nonempty) boundary, to obtain an open manifold N. Then $|K|$, being contractible, cannot separate any points of $N - |K|$. To see that C has a 2-spine L containing K, let $p \in N - M$ and draw arcs from p to the barycenters of each 3-cell of C missing $|K|$; the arcs are routes for collapsing away 3-cells. (Later, in the proof of Theorem 2, we need to invoke the same process, but under the slightly weaker assumption that K has a 1-spine.) But in fact, $K = L$. Since $C^{(1)} \leqslant K$, $L - K$ is some number of open 2-cells; but since $C \searrow L$ and $K \searrow 0$, any such open 2-cell would contribute to $\beta_2(M)$ which is 0.

Thus we only need to exhibit the desired T and K. For later convenience, we shall also insist that the tree T contain all the vertices of C.

We induct on n. If $n = 1$, let T be a maximal tree in $C^{(1)}$ and let K consist of all but one 2-face of σ_1. For $n > 1$, let $\tilde{C}$ be the complex consisting of $\sigma_1, \ldots, \sigma_{n-1}$. We would like to assume, inductively, that appropriate subcomplexes $\tilde{T}$ and $\tilde{K}$ have been constructed and show how to enlarge them to the desired T and K. Unfortunately, $|\tilde{C}|$ may not be a 3-manifold or may not be acyclic. To overcome this problem, we need to be dealing with a slightly more general class of 3-complexes. Theorem 1 follows from Theorem 2 below. □

Theorem 2 (The Special Case). *Let C be a finite simplicial complex, comprised of 3-simplexes, and let $e = \sigma_1, \ldots, \sigma_n$ be an enumeration of the 3-simplexes of C. Then there exist subcomplexes V, T, G, and K of C such that*

(1) $V \leqslant C^{(0)} \leqslant T \leqslant G \leqslant C^{(1)} \leqslant K \leqslant C^{(2)}$,

(2) $K \searrow G$,

(3) *$G - T$ consists of $\Gamma(e)$ open 1-cells,*

(4) $T \searrow V$,

(5) *V consists of one vertex in each component of C.*

Furthermore, if $|C|$ is a proper subpolyhedron of the interior of a 3-manifold, then C has a 2-spine L containing K and $L - K$ consists of $\beta_2(C) - \beta_1(C) + \Gamma(e)$ open 2-cells.

Proof. As in the Very Special Case, if K is any 2-complex containing $C^{(1)}$ such that K has the homotopy type of a graph, assuming $|C|$ is in a 3-manifold guarantees that C has a 2-spine containing K. It is easy to compute, e.g. using Euler characteristic, the number of 2-cells in $L - K$. We proceed to construct V, T, G, and K inductively.

If C is disconnected, then e induces an enumeration e_k of the 3-simplexes of each component C_k of C. Since C_k has fewer 3-cells than C, we may assume that appropriate subcomplexes V_k, T_k, G_k, K_k exist. Summing these gives the desired V, T, G, K, where condition (3) follows from the observation that $\Gamma(e) = \sum_k \Gamma(e_k)$.

Assume now that C is connected, and let $\tilde{C}$ consist of the 3-simplexes with enumeration $\tilde{e} = \sigma_1, \ldots, \sigma_{n-1}$. Inductively, there exist subcomplexes $\tilde{V}$, $\tilde{T}$, $\tilde{G}$, $\tilde{K}$ satisfying (1)-(5) for $\tilde{C}$, $\tilde{e}$. We enlarge $\tilde{T}$, $\tilde{G}$, $\tilde{K}$ in 4 steps as follows:

Step 1. Engulf the vertices of σ_n. Successively add to $\tilde{T}$ (also to $\tilde{G}$, $\tilde{K}$, $\tilde{C}$) just enough of the 1-cells of σ_n to include any vertices not yet covered. Each component of the new $\tilde{T}$ is still a tree, but now $\tilde{T}$ contains $C^{(0)}$. The 1-cells comprising $\tilde{G}-\tilde{T}$ are the same as before and we still have $\tilde{K} \searrow \tilde{G}$.

Step 2. Connect the components of $\tilde{C}$. Since C is connected, each component of $\tilde{C}$ hits σ_n. Successively adjoin to $\tilde{T}$ etc. 1-cells of σ_n connecting different components of $\tilde{C}$. At the conclusion of this step, $\tilde{T}=T$ is in its final form (a connected tree), $\tilde{G}-\tilde{T}$ is the same as before, and $\tilde{K}$ still collapses to $\tilde{G}$.

Step 3. Connect the components of $\tilde{C}\cap\sigma_n$. If any component of the original $\tilde{C}$ hit σ_n in a disconnected set, then $\tilde{C}\cap\sigma_n$ is still disconnected after Step 2; say this set has m components. Successively adjoin to $\tilde{G}$, $\tilde{K}$, $\tilde{C}$ 1-cells of σ_n connecting different components of $\tilde{C}\cap\sigma_n$. With $m-1$ such additions, we obtain $\tilde{G}=G$ satisfying condition (3). Since the same new 1-cells are added to $\tilde{G}$ and to $\tilde{K}$, we still have $\tilde{K} \searrow G$.

Step 4. Engulf the rest of the 1-skeleton of σ_n. So far, $\tilde{K}\cap\sigma_n$ is a connected subcomplex of $\mathrm{bd}(\sigma_n)$ containing all the vertices. We assert that $\tilde{K}$ can be enlarged, by a sequence of elementary simplicial expansions, to a 2-complex K containing all the 1-cells of $\mathrm{bd}(\sigma_n)$. This follows from Lemma 2.1 below. Since $K \searrow \tilde{K} \searrow G$, condition (2) is satisfied and the proof of Theorem 2 is completed. □

Lemma 2. *If S is a cell complex with $|S|$ homeomorphic to a 2-sphere and R is a connected subcomplex of S containing $S^{(0)}$, then either $S^{(1)}\leqslant R$ or there exists a 2-cell τ of S such that R contains all but exactly one edge of* $\mathrm{bd}(\tau)$.

Proof. (Here we do not distinguish $|X|$ from X.) The quotient space S/R is the sum of finitely many 2-spheres joined at one common point $\pi(R)$. If α is a 1-cell of S not contained in R then $\pi(\alpha)$ is a simple closed curve on one of the 2-spheres. If we choose α so that $\pi(\alpha)$ is an innermost such curve then α and some arc in R cobound a disk on S whose interior is disjoint from any 1-cells. This disk is the desired 2-face τ. □

We would like to use sequential connectivity to obtain results such as Lickorish's observation [4] that attaching a disk to a cube with a knotted hole produces a polyhedron that is not collapsible. To do this, we need to generalize Theorem 2 by allowing cells that are not simplexes and principal cells of any dimension $\leqslant 3$. Also, to enable possible application of this machine to studying polyhedra that do not live in 3-manifolds, we state an alternate way to guarantee the existence of a 2-spine L containing K.

Theorem 3 (The General Case). *Let C be a finite cell complex of dimension $\leqslant 3$ and let $e=\sigma_1,\ldots,\sigma_n$ be an enumeration of the principal cells of C. Then there exist subcomplexes V, T, G, and K of C such that*

(1) $V\leqslant C^{(0)}\leqslant T\leqslant G\leqslant C^{(1)}\leqslant K\leqslant C^{(2)}$,

(2) $K \searrow G$,
(3) *$G - T$ consists of $\Gamma(e)$ open 1-cells,*
(4) $T \searrow V$,
(5) *V consists of one vertex in each component of C.*

Furthermore, if $|C|$ is a proper subpolyhedron of the interior of a 3-manifold, or if e has the property that no 3-dimensional cell σ_i has its whole 2-sphere boundary contained in $\bigcup_{j<i} \sigma_j$, then C has a 2-spine L containing K and $L - K$ consists of $\beta_2(C) - \beta_1(C) + \Gamma(e)$ open 2-cells.

Proof. The proof of Theorem 2 needs only slight modification to yield the desired V, T, G, K. We proceed as above through Step 3. In Step 4, we need to consider the additional cases where σ_n is a 1- or 2-cell. If σ_n is a 1-cell, it is already contained in $\tilde{K}$. If σ_n is a 2-cell then either $\tilde{K}$ contains all of $\mathrm{bd}(\sigma_n)$, in which case we take $K = \tilde{K}$, or all but one edge of $\mathrm{bd}(\sigma_n)$, in which case we let $K = \tilde{K} \cup \sigma_n$. Finally, if we are not assuming that $|C|$ embeds in a 3-manifold then we need to construct the 2-spine L along with the other subcomplexes. Inductively, $\tilde{C}$ has a 2-spine $\tilde{L}$ containing $\tilde{K}$. If $\dim(\sigma_n) \leq 2$ then let $L = \tilde{L} \cup \sigma_n$. If $\dim(\sigma_n) = 3$ then, since the expansion of $\tilde{K}$ to K (in Step 4) is stopped as soon as all the edges are engulfed, there is at least one 2-face of σ_n not contained in K. □

Collapsible vs. sequentially connected

We wish to use the idea of sequential connectivity to show that various polyhedra are not collapsible. To do this, we first need to check the extent to which collapsible complexes are sequentially connected (after suitable subdivision perhaps). Some subdivision must be allowed, since, as the following example illustrates, a 3-complex might be cellularly collapsible yet not have sequential connectivity $\Gamma = 0$. The proof of Theorem 4 does not involve subdividing 2-cells, so a collapsible 2-complex does have $\Gamma = 0$. But Example 2 below shows that a collapsible 2-complex (i.e. acyclic and $\Gamma = 0$) need not be sequentially connected in the strong sense of Bing.

Example 1. Let D be the cell-decomposition, consisting of five 2-cells $\{\sigma_1 = (abcde), \sigma_2 = (abcdf), \sigma_3 = (abec), \sigma_4 = (afdec), \sigma_5 = (aebc)\}$, of the 'dunce hat' [5] shown in Fig. 1, and let C be the 3-complex obtained by adding a 3-cell to D so that two adjacent faces of the 3-cell are identified with σ_1 and σ_2.

It is easy to check that C is cellularly collapsible but has sequential connectivity $\Gamma > 0$ since the union of any three of its principal cells collapses to a circle.

Example 2. Let C_0 be the 2-complex obtained from the dunce hat above by deleting the 2-cell σ_1 and adding a new 2-cell $\sigma = (afde)$. Let A_0 be the arc *abcdf.* Note that the complex C_0 is sequentially connected (e.g. list the 2-cells σ, σ_5, σ_4, σ_3, σ_2) and

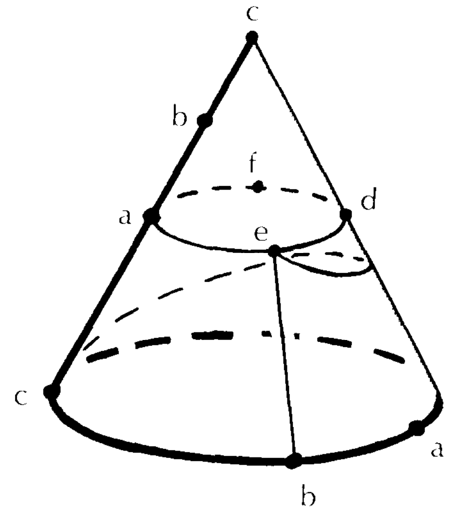

Fig. 1. A certain cell-decomposition of a dunce hat. $\sigma_1 = (abcde)$, $\sigma_2 = (abcdf)$, $\sigma_3 = (abec)$, $\sigma_4 = (afdec)$, $\sigma_5 = (aebc)$.

collapsible, but there is no sequentially connected enumeration that lists σ_2 first. (Any good enumeration must list σ_2 last since it is the only 2-cell whose complement is simply connected.) Also note that σ_2, whose boundary includes the arc A_0, is the only 2-cell of C_0 that meets A_0 in a connected set.

Let C_1 be another copy of C_0 and let A_1 denote the arc corresponding to A_0. To the disjoint union $C_0 \cup C_1$ adjoin a 2-cell $I \times [0, 1]$ so that $I \times \{0\}$ is identified with A_0 and $I \times \{1\}$ with A_1. The resulting 2-complex C is collapsible (hence, by Theorem 4, has an enumeration with $\Gamma = 0$) but not sequentially connected. In enumerating the 2-cells of C, $I \times [0, 1]$ would be a necessary bridge between C_0 and C_1. As soon as $I \times [0, 1]$ were listed, the remaining cells would have to include a sequentially connected enumeration either of C_0 or of C_1 that began with σ_2.

Theorem 4. *If C is a collapsible cell complex of dimension ≤ 3 then C has a subdivision C^* with $\Gamma(C^*) = 0$.*

Remarks. (1) In passing to C^*, no 1- or 2-cells of C are subdivided. The 2nd barycentric subdivision of C, which is a stellar subdivision of our C^*, could be used in place of C^*.

(2) The actual conclusion of the theorem is that C^* has an enumeration satisfying Theorem 3.

Proof. We define C^* as follows: No 1- or 2-cell of C is subdivided; each 3-cell σ is cut into 3-cells σ' and $\{v_\tau * \tau \mid \tau \text{ is a 2-face of } \sigma\}$, where v_τ is a new vertex in

$\text{int}(\sigma)$ near τ, $v_\tau * \tau$ is the join of v_τ with τ, and the v_τ are close enough to the corresponding faces that $v_\tau * \tau \cap v_\mu * \mu = \tau \cap \mu$; σ' is the 'star shaped' closure of the complement of all the $v_\tau * \tau$.

To show that the principal cells of C^* have an enumeration with $\Gamma = 0$ satisfying Theorem 3, we proceed by induction on the number of elementary collapses, $E(C)$, needed to collapse the original complex C. Note that the number $E(C)$ is independent of the particular route of collapsing. Suppose $C \searrow \tilde{C}$ is an elementary collapse (so $C - \tilde{C}$ is an open principal cell σ of C together with a free face of σ) that is the first step in collapsing C; then $\tilde{C} \searrow 0$ and $E(\tilde{C}) < E(C)$. Inductively, the subdivision $\tilde{C}^*$ has an enumeration satisfying Theorem 3.

If C has any 3-cells, we may assume (since collapsing can always be started at the top dimension) that σ is a 3-cell. Each principal cell of $\tilde{C}^*$ either is principal in C^* or is a 2-face τ of σ that is free in C. In enumerating $\tilde{C}^*$, in place of each such 2-face τ, list the 3-cell $v_\tau * \tau$ (which meets the union of its predecessors exactly as τ did). Next append, in any order, the 3-cells $v_\tau * \tau$ for which τ is in $\tilde{C}$ but not principal. Finally, list the star shaped σ' and then $v_\tau * \tau$, where τ is the original free face of σ.

We now consider the case where C has no 3-cells (and so $C^* = C$). The case where $\dim(C) = 1$ is trivial, so we suppose σ is a 2-cell. Let $\bar{C}$ be obtained from $\tilde{C}$ by removing every open edge of $\text{bd}(\sigma)$ that is principal in $\tilde{C}$ and let $K_1, \ldots, K_p$ be the nondegenerate components of $\bar{C}$. Since C has a 1-spine, so does each K_j; since K_j meets the closure of its complement in C in a proper subset of $\text{bd}(\sigma)$, hence a set with no 1-cycles, any nontrivial 1-cycles in K_j would survive in C. Thus K_j is collapsible and, inductively, the principal cells of each K_j (which are principal in C) have an enumeration e_j with $\Gamma(r_j) = 0$. We claim that listing $e_1, \ldots, e_p, \sigma$ has $\Gamma = 0$. Before σ is listed, since the various K_j are pairwise disjoint and $\Gamma(e_j) = 0$, the total is $-(p-1)$. Thus we only need to check that $\sigma \cap (K_1 \cup \cdots \cup K_p)$ has p components. If any $K_j \cap \sigma$ were disconnected, then $H_1(C)$ would be nontrivial. Since the K_j don't meet, their intersections with σ are pairwise disjoint. Thus $\sigma \cap (K_1 \cup \cdots \cup K_p)$ is a collection of p pairwise disjoint arcs and/or vertices.

An application

1. Let Q be a cube with a knotted hole and let C be the space obtained by attaching a 2-cell σ to Q (in particular in a way that kills $\pi_1(Q)$ so C is contractible). It was noted in [4] that C is not polyhedrally collapsible. Here is a proof using sequential connectivity. We first state a lemma that serves a similar purpose to the Corollary on p. 159 of [4].

Lemma 5. *If C is a polyhedron and Q is the space obtained by removing the interior of a principal 2-cell of C, then $\Gamma(C) \leq \Gamma(Q) \leq \Gamma(C) + 1$.*

Remark. To show how to pass from an enumeration for C to an enumeration for Q that increases Γ at most 1, we need (at least this proof does) to be able to subdivide Q. The proof takes a little care and is given at the end of this section.

If C were collapsible, by Theorem 4, we then would have $\Gamma(C^*)=0$. But Lemma 5 would then say that $\Gamma(Q)\leq 1$ and so $\pi_1(Q)$ would be cyclic, contradicting the fact that Q is a cube with a *knotted* hole.

2. If we are interested in attaching several disks to a manifold Q, we can apply Lemma 5 to each one and conclude the following:

Proposition 5. *Let Q be a 3-manifold (or other polyhedron of* $\dim \leq 3$*) such that $\pi_1(Q)$ requires more than n generators. If C is a 3-complex obtained by attaching n 2-cells to Q then $\Gamma(C)>0$ and C is not collapsible.*

Proof of Lemma 5.1. Let σ denote the removed 2-cell. Any enumeration of the principal cells of Q can be followed by σ, which meets Q in its whole boundary and so contributes nothing more to Γ; thus $\Gamma(C)\leq\Gamma(Q)$.

Conversely, suppose we are given an enumeration of principal cells of C that realizes $\Gamma(C)$. If, when σ appears in the list, we have $\sigma\cap(\text{predecessors})=\text{bd}(\sigma)$, then we can omit σ from the list to obtain an enumeration for $\tilde{C}$ with the same Γ.

Assume now that σ meets its predecessors in a proper subset of $\text{bd}(\sigma)$. We wish to insert several cells of Q (that have not yet been listed) into the enumeration in place of σ. Suppose $\sigma\cap(\text{predecessors})$ has p components (so σ contributes $p-1$ to $\Gamma(C)$). Each component of $\text{bd}(\sigma)-(\sigma\cap(\text{predecessors}))$ is an open arc, and there are p of them: $A_1,\ldots,A_p$. (If σ is disjoint from its predecessors then we use any open arc obtained by deleting one vertex from $\text{bd}(\sigma)$.) We shall handle each A_j separately, and the order is immaterial, so let A be any one of the A_j.

Orient A and let $\alpha_1,\ldots,\alpha_m$ be, in order, the closed 1-cells of $\text{bd}(\sigma)$ whose interiors lie in A. We shall insert a list of principal cells of (possibly subdivided) Q, $\{\alpha'_1,\ldots,\alpha'_m\}$, where the α'_j are defined as follows: If α_j is principal in Q then let $\alpha'_j=\alpha_j$. If α_j is not principal in Q, then α_j must be an edge of a principal cell of C that appears later than σ in the enumeration of C. Let β be the first such cell and let α'_j be the closed star of $\text{int}(\alpha_j)$ in the second barycentric subdivision of β (i.e. a regular neighborhood of α_j in β pinched off at the endpoints). [Later in the enumeration, when it's time to list β, we list the slightly smaller $\beta'=\text{clos}(\beta-\alpha')$.] The α'_j meet like arcs strung end-to-end and so contribute 0 to $\Gamma(Q)$, except for α'_m which meets its predecessors at both endpoints of α_m and so contributes $+1$ to Γ. Since each of the p arcs $A_1,\ldots$ causes one such contribution, we have a net increase of $+1$ over the contribution originally made by σ.

After listing all the α'_j from all the arcs A, whatever portion of $\text{bd}(\sigma)$ that was not covered by $\sigma\cap(\text{predecessors})$ has now been covered. Thus we can resume the enumeration of C with each additional cell making the same contribution to $\Gamma(Q)$ as it did to $\Gamma(C)$. $\square$

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